Online Appendix

To use our package named `ConvergenceConcepts`, download it from CRAN (`http://cran.r-project.org`) and install it along with the required dependencies. Then launch R and type in its console the following instructions:

```
require(ConvergenceConcepts)

investigate()
```

Now, you can investigate all the examples and exercises presented in this paper.

A Examples

A.1 Convergence in r-th mean

Example 1. We would like to investigate the convergence in r-th mean (for $r = 1, 2, 3$ say) of X_n towards $X = 0$, where the X_n are independent random variables such that $P[X_n = n^{0.4}] = 1/n$ and $P[X_n = 0] = 1 - 1/n$. One can show that $E|X_n|^r = n^{0.4r-1}$ and thus $X_n \xrightarrow{r} 0$ for $r = 1, 2$ but not for $r = 3$. This can be observed on the following plot (see Figure 1) where we took $nmax = 2000$ and $M = 500$.

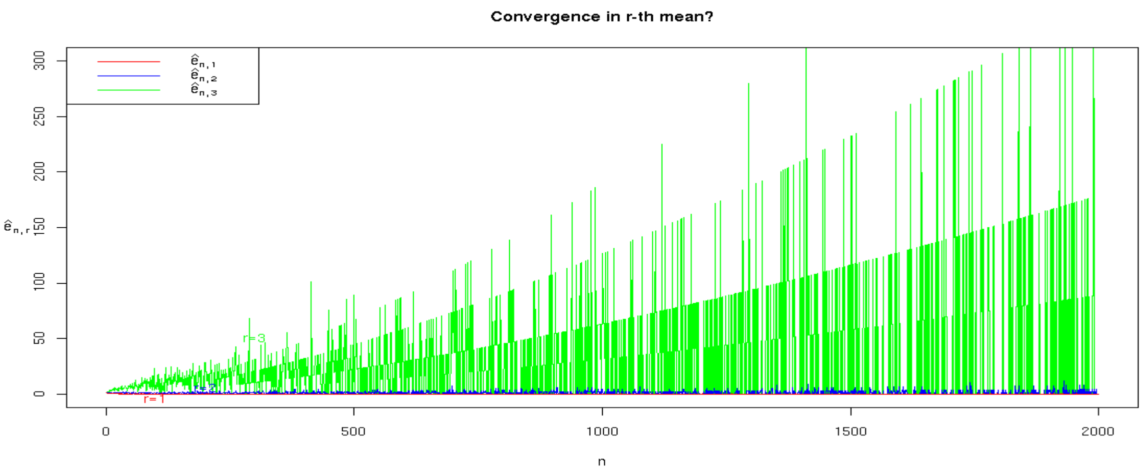

Figure 1: $\hat{e}_{n,1}$ (red) and $\hat{e}_{n,2}$ (blue) going towards 0 ; $\hat{e}_{n,3}$ (green) not going towards 0.

A.2 Convergence in law

Example 2. Figure 2 shows the convergence in distribution of $X_n = \frac{1}{\sqrt{n}}\left[\frac{\sum_i^n Z_i - n}{\sqrt{2}}\right]$ towards $N(0, 1)$ where the Z_i are *i.i.d.* χ^2_1 random variables. On the left you can see an output of our `law.plot2d` function with the slider value fixed at $n = 70$. The distribution function of a standard Gaussian is plotted in black whereas the empirical distribution function of X_n based on $M = 5000$ realizations is plotted in red. We can move the slider and see that the red curve comes closer to the black one. Also, on the right you can see the tri-dimensional plot of $|\hat{F}_n(t) - F(t)|$ for $n = 1, \ldots, nmax = 200$ to see if gets closer to the zero horizontal plane. These plots suggest a convergence in distribution.

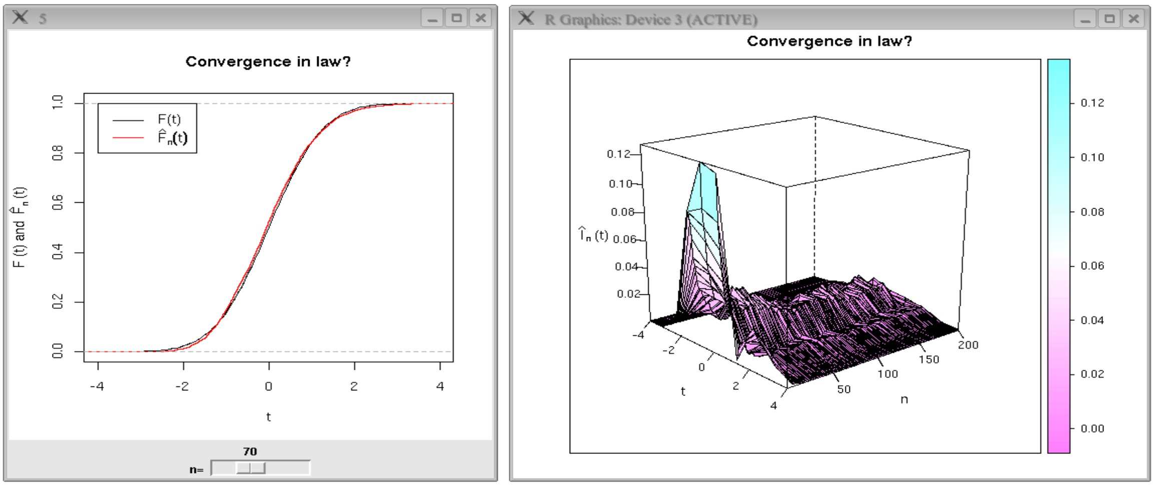

Figure 2: Convergence in distribution in action on a simulated example. Left: the distribution function of a standard Gaussian is plotted in black whereas the empirical distribution function of X_n ($n = 70$) based on $M = 5000$ realizations is plotted in red. Right: tri-dimensional plot of $|\hat{F}_n(t) - F(t)|$ as a function of n and t.

B Exercises

Exercise 1. Let $X_1, X_2, \ldots, X_n$ be *i.i.d.* $N(0,1)$ random variables and $X = X_1$. Does $X_n \xrightarrow{L} X$? Does $X_n \xrightarrow{P} X$?

Exercise 2. Let $X_1, X_2, \ldots, X_n$ be independent random variables such that $P[X_n = \sqrt{n}] = \frac{1}{n}$ and $P[X_n = 0] = 1 - \frac{1}{n}$. Does $X_n \xrightarrow{2} 0$? Does $X_n \xrightarrow{P} 0$?

Exercise 3. Let Z be $U[0,1]$ and let $X_n = 2^n 1_{[0,1/n)}(Z)$. Does $X_n \xrightarrow{r} 0$? Does $X_n \xrightarrow{a.s.} 0$?

Exercise 4. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with mean 0 and variance 1. Define $X_1 = X_2 = 1$ and

$$X_n = \frac{\sum_{i=1}^n Y_i}{(2n \log\log n)^{1/2}}, n \geq 3.$$

Does $X_n \xrightarrow{2} 0$? Does $X_n \xrightarrow{a.s.} 0$?

Exercise 5. Let $Y_1, Y_2, \ldots, Y_n$ be independent random variables with uniform discrete distribution on $\{0, \ldots, 9\}$. Define

$$X_n = \sum_{i=1}^n \frac{Y_i}{10^i}.$$

It can be proved that $X_n \xrightarrow{a.s.} X = \sum_{i=1}^\infty \frac{Y_i}{10^i}$ which follows a $U[0,1]$ distribution. Now, let $Z \sim U[0,1]$ independent of X.

Does $X_n \xrightarrow{a.s.} Z$? Does $X_n \xrightarrow{L} Z$?

Solution to Exercise 1.

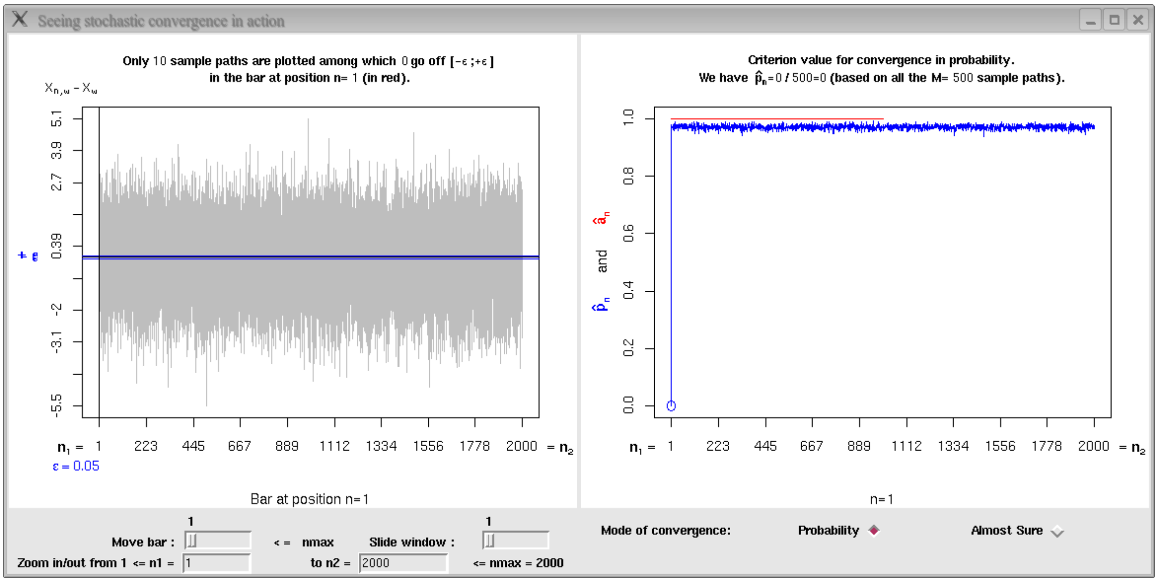

Figure 3: Ten sample paths of $X_n - X_1$ amid the 500 (left); $\hat{p}_n$ (resp. $\hat{a}_n$) going towards $p_n \neq 0$ (resp. $a_n = 1$) (right).

It is trivial that X_n converges in law to X_1 since for each n both X_n and X have the same distribution function. Now, since X_n and X are independent, $X_{n,\omega} - X_\omega$ has no particular reason to be close to 0 for any n or any ω. Thus we do not have $X_n \xrightarrow{P} X$. It can be seen on the plot of Figure 3 that $X_{n,\omega} - X_\omega$ tends to be far from 0 and that $\hat{p}_n$ and $\hat{a}_n$ are not going towards 0. Indeed, in this case, by noting that $X_n - X \sim N(0, 2)$, one can obtain explicitly

$$p_n = 2\left[1 - \Phi\left(\epsilon/\sqrt{2}\right)\right] \simeq 0.9718 \neq 0 \text{ (for } \epsilon = 0.05) \tag{1}$$

where $\Phi(\cdot)$ denotes the standard $N(0, 1)$ distribution function. Thus $X_n \overset{a.s.}{\nrightarrow} X$ and $X_n \overset{P}{\nrightarrow} X$.

Solution to Exercise 2.

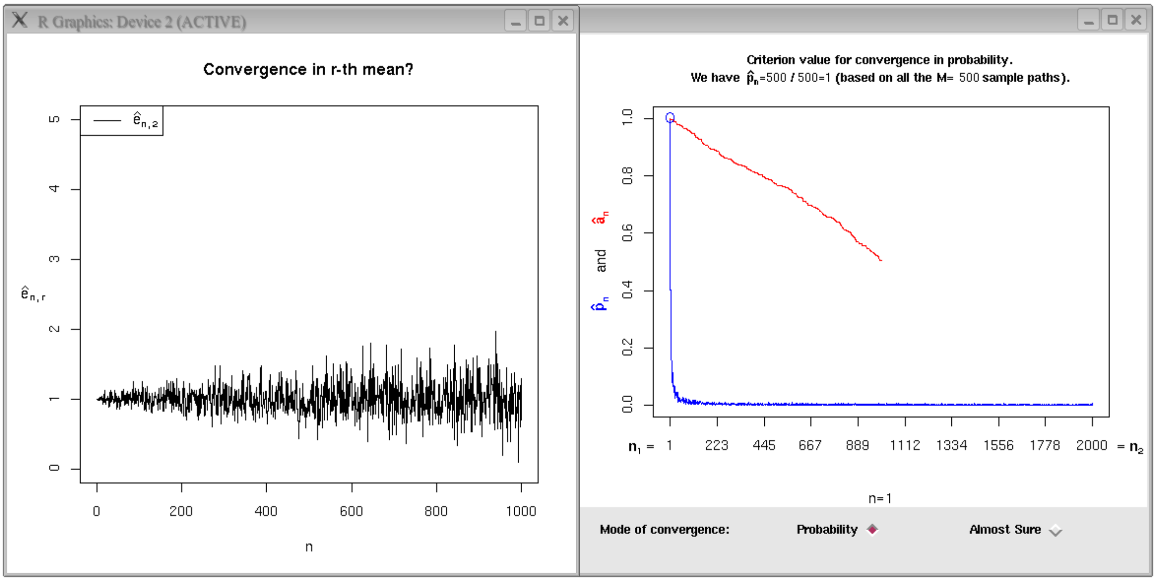

Figure 4: $\hat{e}_{n,2}$ not going towards 0 (left) and $\hat{p}_n$ going towards 0 (right).

We can mentally visualize each sample path to be essentially equal to 0, but to sometimes jump higher and higher, as n increases, with a decreasing probability. This gives us the intuition that X_n converges in probability to 0. On the other hand, for a fixed n, the mean of the $X_{n,\omega}^2$ values is taken away from 0 due to these few but very large values. But for increasing values of n, one can not say if the mean of the $X_{n,\omega}^2$ values will decrease or not. So we cannot tell more about the quadratic mean convergence to 0.

The intuition for convergence in probability is confirmed using our package ($\hat{p}_n$ is going to 0, see Figure 4). But we can expect that we do not have convergence in quadratic mean towards 0 because $\hat{e}_{n,2}$ is not going to 0 but oscillates around 1.

Now, one can prove that X_n does not converge in a quadratic mean to 0 since $e_{n,2} = E|X_n|^2 = 1, \forall n$ and that X_n converges to 0 in probability since $p_n = \frac{1}{n} \to 0$.

Solution to Exercise 3.

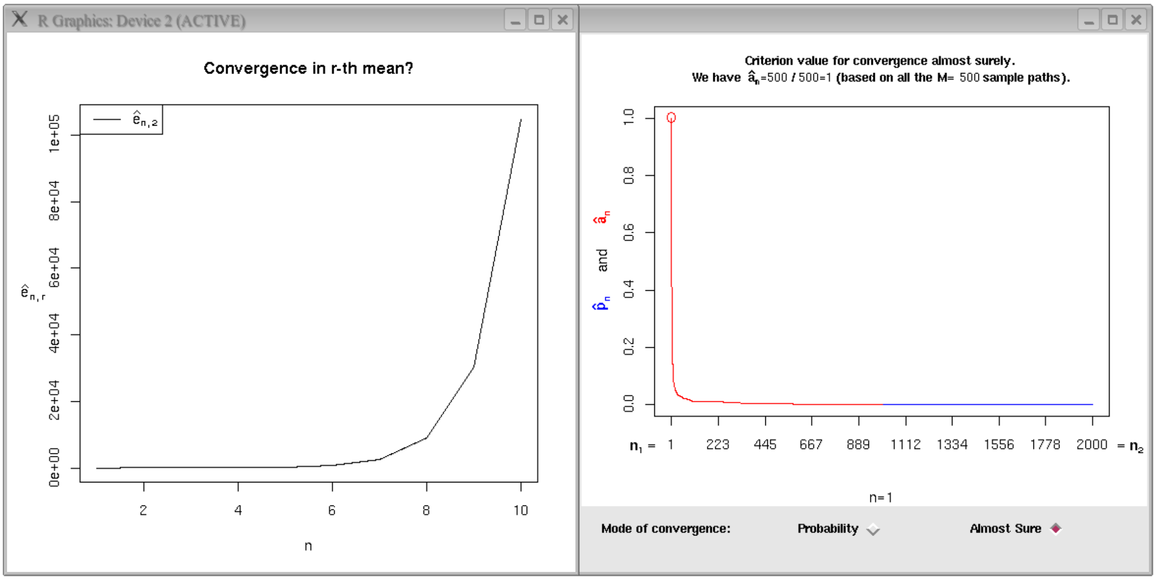

Figure 5: $\hat{e}_{n,2}$ not going towards 0 (left) and $\hat{a}_n$ going towards 0 (right). We plotted the left graph only for the very first n values since divergence is very fast here.

We can mentally visualize each sample path to be growing to large values then suddenly dropping to 0 and after that staying infinitely at this null value. These sample paths can also be visualized using our package with the possibility to use the "zoom in" facility. This gives us the intuition that X_n converges almost surely to 0 since $\forall\omega$, $\lim_{n\to\infty} X_{n,\omega} = 0$. On the other hand, for a fixed n, the mean of the $X^2_{n,\omega}$ values is taken away from 0 due to the small proportion of sample paths that take very large values. But for increasing values of n, one can not say if the mean of the $X^2_{n,\omega}$ values will decrease or not. So we cannot tell more using our intuition about the quadratic mean convergence to 0.

Convergence almost surely to 0 is illustrated using our package ($\hat{a}_n$ is going to 0, see Figure 5). But we can expect that we do not have convergence in quadratic mean towards 0 because $\hat{e}_{n,2}$ is not going to 0.

We can now prove that X_n does not converge to 0 in r-th mean since $E|X_n|^r = \frac{2^{rn}}{n} \to \infty$.

Solution to Exercise 4.

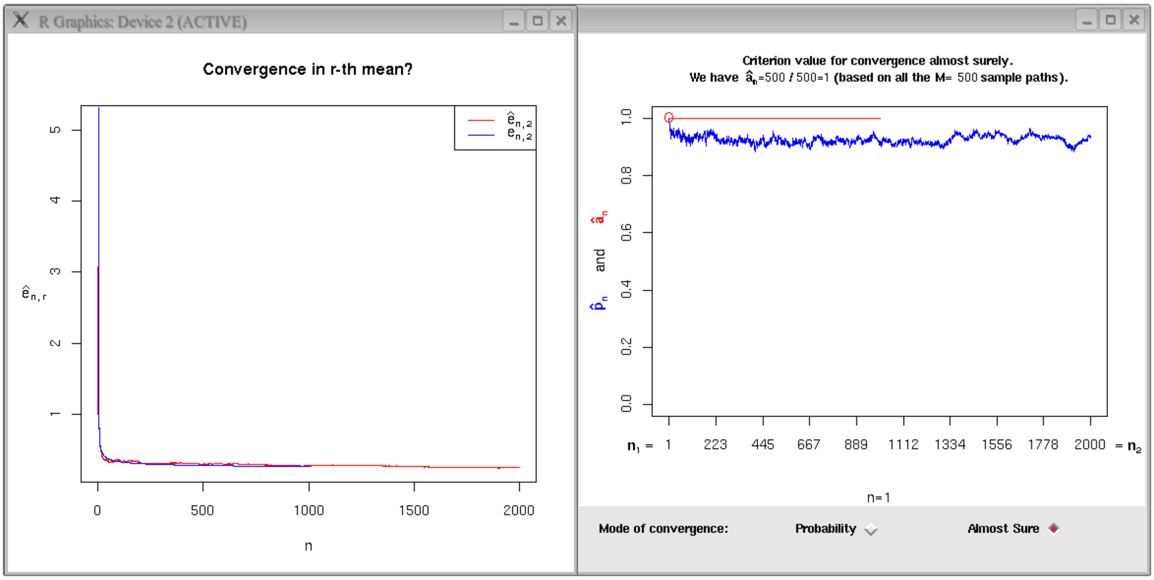

Figure 6: $\hat{e}_{n,2}$ going towards 0 (left) and $\hat{a}_n$ equals 1 (right).

Looking at the definition of X_n, we do not get a precise information on the sample paths. So, intuition cannot be of great help in this case. Thus, we use our package (with Y_i i.i.d. $N(0,1)$) to get some clue on quadratic convergence and almost sure convergence.

Figure 6 shows that $\hat{e}_{n,2}$ is going towards 0 and that $\hat{a}_n$ equals 1. This suggests a quadratic mean convergence, and not an almost sure convergence.

We can now prove that X_n converges in a quadratic mean to 0 since $E|X_n|^2 = \frac{1}{2 \log \log n}$ for all n. We added a blue curve on the plot for the function $e_{n,2} = \frac{1}{2 \log \log n}$ and we see that the blue and red curves are superposed.

To prove almost sure convergence, we have to use the law of the iterated logarithm (see Billingsley, 1995, p.154) that can be formulated as $P[X_n > 1 - \epsilon, \text{ infinitely often}] = 1$. This suffices to prove that X_n does not converge to 0 almost surely.

Solution to Exercise 5.

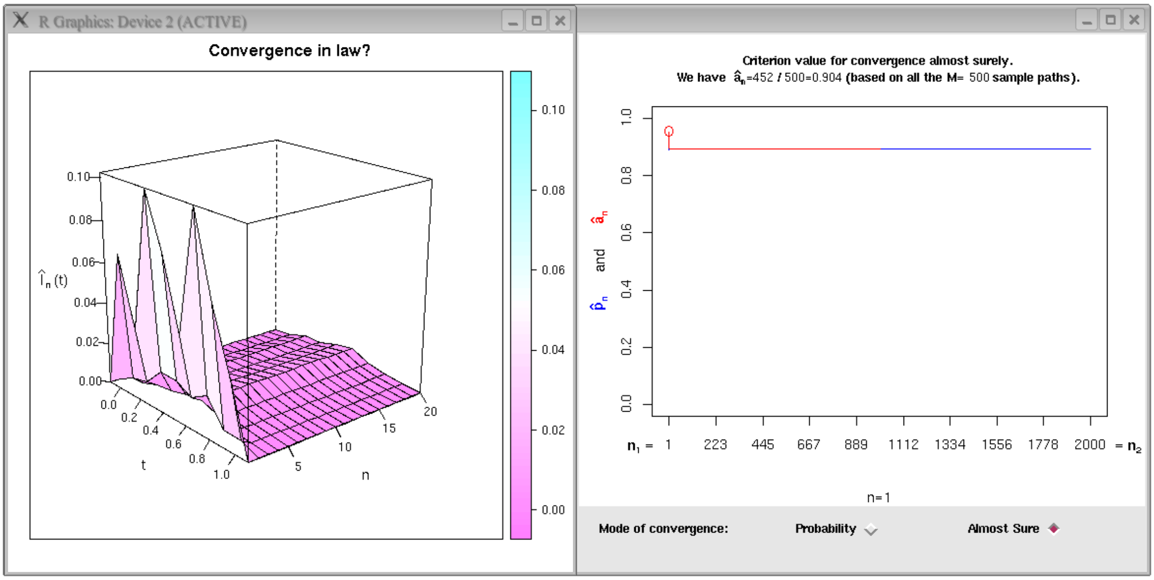

Figure 7: $\hat{l}_n(t)$ going towards 0 (left) ; $\hat{a}_n$ not going to 0 (right).

Since X_n and Z are independent, $X_{n,\omega} - Z_\omega$ has no particular reason to be close to 0 for any n or any ω. Thus we do not have $X_n \xrightarrow{a.s.} Z$. It can be seen on the plot of Figure 7 that $X_{n,\omega} - Z_\omega$ tends to be far from 0 and that $\hat{a}_n$ is not going towards 0. Using our package, we can also see that $\hat{l}_n(t)$ is going towards 0 forall t. This suggests a convergence in law of X_n towards Z. Indeed, as almost sure convergence implies convergence in law, we have $X_n \xrightarrow{L} X$ and since X and Z are both $U[0,1]$, $X_n \xrightarrow{L} Z$.

Now, lets us prove rigorously that $X_n \overset{a.s.}{\nrightarrow} Z$. We have $X_n - Z = X_n - X + X - Z \xrightarrow{a.s.} X - Z$ (by Slutsky theorem, see Ferguson (1996) p.42). Therefore $X_n - Z \xrightarrow{L} X - Z$ which implies that $\forall \epsilon > 0, p_n = P[|X_n - Z| > \epsilon] \underset{n\to\infty}{\longrightarrow} P[|X - Z| > \epsilon] = (1-\epsilon)^2 = 0.9025$ (for $\epsilon = 0.05$). Thus, $X_n \overset{P}{\nrightarrow} Z$ and so $X_n \overset{a.s.}{\nrightarrow} Z$. Note that the density function $p(.)$ of the difference of two $U[0,1]$ is given by $p(z) = (1+z)1_{\{-1\leq z\leq 0\}} + (1-z)1_{\{0\leq z\leq 1\}}$.

Note that if X and Z are two independent non constant random variables with the same law, we can have $X_n \xrightarrow{a.s.} X$ (i.e. $X_{n,\omega} \to X_\omega$ almost everywhere) but $X_n \overset{a.s.}{\nrightarrow} Z$ because we may not have $X_\omega = Z_\omega$ $\forall \omega \in \Omega$. But, in the case where X and Z are constant random variables with the same law, we have obviously $X = Z$ and thus trivially $X_n \xrightarrow{a.s.} X$ implies that $X_n \xrightarrow{a.s.} Z$.